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A discrete-space urban model with environmental amenities

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Abstract

This paper analyzes the effects of providing environmental amenities associated with open space in a discrete-space urban model and characterizes optimal provision of open space across a metropolitan area. The discrete-space model assumes distinct neighborhoods in which developable land is homogeneous within a neighborhood but heterogeneous across neighborhoods. Open space provides environmental amenities within the neighborhood it is located and may provide amenities in other neighborhoods (amenity spillover). We solve for equilibrium under various assumptions about amenity spillover effects and transportation costs in both open-city (with in- and out-migration) and closed-city (fixed population) versions of the model. Increasing open space tends to increase equilibrium housing density and price within a neighborhood. In an open-city model, open space provision also increases housing density and price in other neighborhoods if there is an amenity spillover effect. In a closed-city model, housing density and prices in other neighborhoods can decrease if the pull of the local amenity value is stronger than the push from reduced availability of developable land. We use numerical simulation to solve for the optimal pattern of open space in two examples: a simple symmetric case and a simulation based on the Twin Cities Metropolitan Area, Minnesota, USA. With no amenity spillover, it is optimal to provide the same amount of open space in all neighborhoods regardless of transportation cost. With amenity spillover effects and relatively high transportation cost, it is optimal to provide open space in a greenbelt at the edge of the city. With low transportation cost, open space is provided throughout the city with the exception of neighborhoods on the periphery of the city, where the majority of the population lives. A greenbelt still occurs but its location is inside the city.

Keywords: Open space; Environmental amenities; Discrete urban economics model

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1. Introduction

Metropolitan areas in the US are experiencing rapid growth and large-scale conversion of undeveloped to developed land. Many residents are concerned about the resulting loss of open space and environmental amenities. Some local governments, as well as private land trusts, have instituted policies to acquire land or conservation easements to preserve undeveloped land within or on the fringe of metropolitan areas. From 1996 through 2004, voters approved 1062 of 1373 referenda for open space and parks authorizing the use of $26.4 billion (2000 constant dollars) to acquire open space or development rights (Nelson et al., 2007; Trust for Public Land, 2004).

There are at least two important effects of conserving open space in a metropolitan area. First, open space generates amenities that make nearby areas more attractive, thereby changing the spatial pattern of demand for development. Open space designation may result in shifts in demand between different locales within a given metropolitan area, and it may shift overall demand by encouraging immigration to (or emigration from) the metropolitan area. Second, open space designation restricts the supply of land available for development. For reasons of both demand and supply, the provision of open space affects equilibrium patterns of land prices and density of development within a metropolitan area.

We analyze the effect of designating open space on the spatial pattern of residential development, population, and property values in a discrete-space urban economics model. We divide the city into discrete neighborhoods and assume that developable land is homogeneous within a neighborhood but heterogeneous across neighborhoods. Neighborhoods can differ with respect to the area available for development, the area of open space, access to employment opportunities, and existing environmental amenities. Provision of open space in a neighborhood reduces the area available for development and increases environmental amenities. We consider cases where an open space amenity only affects the neighborhood in which it is located (local public good) and where an amenity affects multiple neighborhoods (amenity spillover). In the model, landowners choose to rent land to households, the government for open space, or to the agricultural sector. Households maximize their utility by choosing where to live and how much of their income to spend on housing versus other goods. The government chooses property tax rates and the provision of open space given that it must balance its budget.

We analyze equilibrium outcomes for both an open-city model in which population adjusts so that utility is the same within the metropolitan area as elsewhere, and a closed-city model in which population is held constant and utility levels vary. In both open-city and closed-city models we show that equilibrium housing density and after-tax land price in a neighborhood tend to increase with open space provided in that neighborhood. In an open-city model, open space provision increases housing density and price in other neighborhoods as well if there is an amenity spillover effect. In a closed-city model, whether housing density and prices in other neighborhoods increase or decrease depends on whether the push from reduced availability of developable land in the neighborhood with increased open space outweighs the pull of the local amenity value in that neighborhood. For an open city, we also show that the incidence of the property tax falls solely on landowners and how taxes are raised (e.g., a uniform city-wide tax or a neighborhood-specific tax) does not affect equilibrium outcomes as long as the city boundaries do not change.

In addition to analyze equilibrium housing densities and land prices for a given pattern of open space, we formulate and numerically solve the problem of determining the optimal size and location of open space using an open-city model. The presence of amenity spillover has a strong effect on the results. When open space is a local public good that affects only the immediate neighborhood, and assuming Cobb–Douglas preferences, it is optimal to provide the same amount of open space in all neighborhoods. With amenity spillover effects, optimal open space provision differs across neighborhoods. In this case, it is optimal to provide open space in a greenbelt at the edge of the city when transportation cost is relatively high. For low transportation cost, however, open space is provided throughout the city with the exception of neighborhoods on the periphery of the city. A greenbelt still occurs but its location is inside the city.

Modeling the urban area as discrete neighborhoods allows us to more fully develop the analysis of open space amenities. Realistic features such as multiple business centers, existing environmental amenities, and amenity values of agricultural land are incorporated in a closed-form analytic solution for equilibrium housing and land prices. In addition, these features are easily incorporated into the open-city optimization model to determine their impact on the optimal pattern of open space, housing, and land prices. There is also a sense in which neighborhoods, rather than points in space, are the natural unit of analysis. Data often come in neighborhoods units (e.g., census blocks) and neighborhoods provide an important sense of identity within metropolitan areas.

Our discrete-space model contrasts with most urban economics models that utilize continuous-space formulations (see Anas et al., 1998; Huriot and Thisse, 2000 for surveys). In the monocentric city model developed by Alonso (1964), Mills (1967), Mills (1972) and Muth (1969), areas close to the central business district (CBD) are more desirable because of lower commuting cost. These areas have higher land prices and greater housing density. In many urban economics models, locations are identical except for distance to the CBD, i.e., development occurs on a featureless plain. Polinsky and Shavell (1976) include an environmental amenity characterized by its distance to the CBD and show how the amenity changes the spatial pattern of property values. Brueckner et al. (1999) include amenities characterized by distance to the CBD to determine the locations of different income classes. In these two papers, the environmental amenity does not occupy space. In contrast, Mills (1981), Nelson (1985), and Lee and Fujita (1997), analyze the effects of greenbelts that form a ring of open space, which occupy space and are characterized by their distance to the CBD. Lee and Fujita (1997) analyze the optimal placement of a greenbelt. Franco and Kaffine (2005) analyze optimal placement and size of public goods in a single-dimensional model. Several papers develop two-dimensional urban models with environmental amenities that show the effect of the location, size and shape of open space on equilibrium housing, land price, and city boundary in an open-city model (Wu and Plantinga, 2003; Wu, 2006; Kovacs and Larson, 2007). With the exception of Lee and Fujita (1997) and Franco and Kaffine (2005), these papers do not analyze the optimal pattern of open space provision.

A related literature analyzes the optimal allocation of public goods among different locations administered by different taxing authorities. Flatters et al. (1974) analyze the provision of local public good in two regions where labor can migrate between regions. Because a migrant does not account for the fiscal externality of their move, the distribution of the population among regions is optimal only under very special circumstances. Stiglitz (1977), Stiglitz (1983) and Fujita (1989) also analyze the provision of a local public goods given fiscal effects. Berliant et al. (2006) develop a model where the number and location of facilities that provide congestible local public goods are determined endogenously. An important limitation of these models from the standpoint of analyzing open space amenities is that public goods do not take up space and therefore do not compete with housing for land.

Our paper builds on a remarkable paper by Yang and Fujita (1983) that solves for equilibrium housing density and land prices given the provision of open space and solves for the optimal pattern of open space in both open and closed-city models. They use a one-dimensional formulation and show that the optimal density of open space is a uniform proportion of area independent of distance from the CBD when environmental amenities are purely local (i.e., no amenity spillover effects). They briefly consider the case with amenity spillover in a one-dimensional discrete-space model with five neighborhoods. In this paper, we extend the discrete-space model of Yang and Fujita (1983) to a two-dimensional formulation with multiple neighborhoods that includes the amount of open space and housing in each neighborhood. This expansion allows us to consider a richer set of examples with more complicated patterns of optimal open space provision, preexisting amenities and multiple CBDs, and to explore the effects of changes in transportation costs and amenity spillovers.

In the next section we describe the basic discrete-space urban model with open space and other amenities. We define market equilibrium and show how provision of open space in a neighborhood affects housing density and land price within and outside the neighborhood. We then define a social planner's problem to determine the optimal amount and location of open space and numerically solve the problem for two examples. The first example is a symmetric city with a single central business district and no preexisting amenities. In this example we show how the amenity spillover effect and transportation cost affect optimal city size and spatial patterns of open space and housing. The second example is based on data from the Twin Cities Metropolitan Area (Minneapolis–St. Paul, MN, USA), and we show how the presence of two CBDs and spatially heterogeneous existing amenities, including amenities associated with agricultural land, affect the optimal size and location of open space.

2. A discrete-space urban model with open space amenities

In this section we present a spatially explicit model of a city with open space and other environmental amenities. We consider both open-city and closed-city variants of the model. In an open-city model, population is determined endogenously by in- and out-migration. In equilibrium, city residents are equally well off living in the city as elsewhere and utility levels are fixed. In a closed-city model, population is fixed but utility levels vary. The city consists of a set of discrete neighborhoods, Θ, located on an XY coordinate plane. The location of each neighborhood is expressed by its coordinates (x, y), which is the centroid of the neighborhood. A neighborhood's total land area is denoted by $l(x, y)$.

The model specification allows for multiple business centers. This extension of the monocentric city model is motivated by the observation that the central business district is not the sole employment center, nor even the dominant employment center, in many cities. There are J business centers, where J is a positive integer bounded above by the total number of neighborhoods in the city. The business centers are dimensionless. The residents of the city choose to commute to the business center that is located closest to their neighborhood of residence. Let $d_j(x, y) = d((x - x_j), (y - y_j))$ be the Euclidian distance from neighborhood (x, y) to neighborhood (x_j, y_j). We define the commuting distance for people living in neighborhood (x, y) to be

$$d_C(x, y) = \min\,\{d_j(x, y)\}_{j=1}^{J}.$$

The commuting cost to work for a resident living in neighborhood (x, y) is denoted by

$$f[d_C(x, y)],$$

where f is an increasing function of commuting distance, $d_C(x, y)$.

In each neighborhood in the city, land is allocated for residential use and open space. The proportion of land devoted to open space in the (x, y) neighborhood is denoted by $a(x, y)$. Open space creates an environmental amenity that contributes to the well-being of city residents. When there is no amenity spillover effect, open space only contributes to well-being of residents in the neighborhood in which it is located (i.e., it is a local public good). However, with spillover effects, open space contributes to the well-being of residents living in other neighborhoods as well, with the contribution declining with the distance between the neighborhood of residence and the neighborhood in which the open space is located. In addition to the amenity from open space there may also be amenities associated with preexisting features of a neighborhood. These preexisting features vary from natural environmental amenities (e.g., topography, lakes), positive man-made amenities (e.g., schools, theatres), to negative features or disamenities (e.g., waste sites, smokestacks). Denote the proportion of area covered by a preexisting amenity located in neighborhood (x, y) by $z(x, y)$. Similar to open space amenities, preexisting amenities may be either a local public good (or local public bad in the case of a disamenity) or have spillover effects in other neighborhoods.

Land outside of the city is allocated to a non-development use (agriculture). The residents of the city may derive an amenity or a disamenity from their proximity to agricultural land. Denote the proportion of area covered by agricultural land at location (x, y) as $g(x, y)$. Let Ω represent the set of agricultural districts that might contribute amenity value to some neighborhood in the city.

For simplicity, we assume that all land is owned by absentee landowners. Landowners rent land to city residents for housing. Let $p(x, y)$ represent the (pre-tax) residential land rental price in neighborhood (x, y). To provide open space in neighborhood (x, y) the city government rents land from landowners at price $p(x, y)$. Landowners can also rent land to farmers for agriculture. The agricultural rental price is denoted by $p_g(x, y)$. The model endogenously determines the boundary of the city. A neighborhood will be included in the city if and only if $p(x, y) \geq p_g(x, y)$.

The city government collects property tax $\tau(x, y)$ in order to pay for open space provision. The government must satisfy a budget constraint that open space expenditure equals property tax revenue. One version of the budget constraint is that each neighborhood pays for open space provided in the neighborhood, that is

$$\tau(x, y)\, p(x, y)[1 - a(x, y) - z(x, y)]l(x, y) = p(x, y)a(x, y)l(x, y). \tag{1}$$

In this case, the neighborhood property tax rate is: $\tau(x, y) = a(x, y)/(1 - a(x, y) - z(x, y))$. The budget constraint can also be specified as a city-wide constraint with a uniform property tax rate τ:

$$\sum_{(x,y)\,\in\,\Theta} \{\tau\, p(x, y)[1 - a(x, y) - z(x, y)]l(x, y)\} = \sum_{(x,y)\,\in\,\Theta} \{p(x, y)a(x, y)l(x, y)\}. \tag{2}$$

In this case, the uniform city-wide property tax rate is: $\tau = \sum_{(x,y)\,\in\,\Theta}\{p(x, y)a(x, y)l(x, y)\}/\sum_{(x,y)\,\in\,\Theta}\{p(x, y)[1 - a(x, y) - z(x, y)]l(x, y)\}$. One could also write down intermediate cases to cover situations with multiple taxing zones or independent jurisdictions within the metropolitan area.

We assume that all households have identical income v and identical utility function u in consumption, housing, and amenities. Given the distribution of open space, the objective of each household i is to choose a neighborhood of residence $(x, y) \in \Theta$, an amount of residential land $h_i(x, y)$, and consumption good $c_i(x, y)$, to maximize utility subject to its budget constraint. The attractiveness of living in a neighborhood depends upon the commuting costs from the neighborhood to the nearest employment center and the amenities. We define the amenity function for neighborhood (x, y) as

$$A(x, y) = A((x, y), \{a(s, r)\}_{(s,r) \in \Theta}, \{z(s, r)\}_{(s,r) \in \Theta}, \{g(s, r)\}_{(s,r) \in \Omega}), \tag{3}$$

which in general depends on the pattern of the provision of open space across neighborhoods, $\{a(s, r)\}_{(s,r) \in \Theta}$, preexisting amenities across neighborhoods, $\{z(s, r)\}_{(s,r) \in \Theta}$, and amenities from agricultural land outside the city, $\{g(s, r)\}_{(s,r) \in \Omega}$. We assume that the environmental amenity value for neighborhood (x, y) is increasing in open space provided within (x, y): $\partial A(x, y)/\partial a(x, y) > 0$, and that the environmental amenity value for neighborhood (x, y) is non-decreasing in open space in other neighborhoods $\partial A(x, y)/\partial a(s, r) \geq 0$ (strictly increasing with positive spillover effects). We further assume that ''own-neighborhood'' amenity effect is larger than the ''cross-neighborhood'' amenity effect for an increase in open space: $\partial A(x, y)/\partial a(x, y) > \partial A(s, r)/\partial a(x, y)$ for all $(x, y) \in \Theta$, $(s, r) \in \Theta$, $s \neq x$, or $r \neq y$. Preexisting amenities and agricultural amenities can have positive or negative effects. A household i residing in neighborhood (x, y) solves the following problem:

$$\max_{c_i(x,y), h_i(x,y)} u[c_i(x, y), h_i(x, y), A(x, y)] \tag{4a}$$

such that:

$$c_i(x, y) + p^\tau(x, y)h_i(x, y) + f[d_C(x, y)] \leq v \tag{4b}$$

$$c_i(x, y) \geq 0, \qquad h_i(x, y) \geq 0, \tag{4c}$$

where u is a continuous, quasi-concave utility function increasing in each argument and $p^\tau(x, y) = [1 + \tau(x, y)]p(x, y)$ is the after-tax residential land rental price. Note that $\tau(x, y) = \tau$ in the case of a uniform city-wide tax.

3. Market equilibrium

We formulate a market equilibrium for the households, landowners, and the government. Let $n(x, y)$ be the number of households living in neighborhood (x, y). Let $\bar{N}$ be a fixed population size of the city in a closed-city model (N endogenously determined in an open-city model) and let $\bar{u}$ be a fixed reservation utility in an open-city model (u endogenously determined in a closed-city model).

3.1. Definition

Given an open space allocation, preexisting amenities, property taxes, and agricultural rental prices, $\{a(x, y), z(x, y), \tau(x, y), p_g(x, y)\}_{(x,y) \in \Theta}$, location of business centers $\{b(x_j, y_j)\}_{j=1}^{J}$, a uniform utility level $\bar{u}$ in an open city, and a population size $\bar{N}$ in a closed city, a market equilibrium is defined as allocation $\{\{\hat{c}_i(x, y), \hat{h}_i(x, y)\}_{i=1}^{\hat{n}(x,y)}, \hat{n}(x, y)\}_{(x,y) \in \Theta}$ and a price system $\{\hat{p}^\tau(x, y)\}$, such that:

1. Households maximize utility in each location: allocation $\{\hat{c}_i(x, y), \hat{h}_i(x, y)\}_{i=1}^{\hat{n}(x,y)}$ for all $(x, y) \in \Theta$ maximizes utility function (Eq. (4a)) subject to budget constraint (Eq. (4b)) and non-negativity conditions (4c).
2. No arbitrage across locations holds for all neighborhoods $(x, y) \in \Theta$:

$$u[\hat{c}_i(x, y), \hat{h}_i(x, y), A(x, y)] = \bar{u} \text{ in an open city} \tag{5a}$$

and

$$u[\hat{c}_i(x,y),\hat{h}_i(x,y),A(x,y)] = u[\hat{c}_i(s,r),\hat{h}_i(s,r),A(s,r)] \quad \text{for all } (s,r)\in\Theta \text{ in a closed city.} \tag{5b}$$

3. Determination of city boundaries satisfies:

$$\hat{p}(x,y) \geq p_g(x,y) \quad \text{for each neighborhood } (x,y)\in\Theta \quad \text{and} \quad \hat{p}(x,y) < p_g(x,y) \quad \text{for any location } (x,y)\notin\Theta. \tag{6}$$

4. The government balances its budget: for each neighborhood $(x,y)\in\Theta$ with the neighborhood tax (Eq. (1)), or with the city-wide tax (Eq. (2)).
5. The land market clears for all neighborhoods $(x,y)\in\Theta$ (i.e., residential area, plus open space area, plus preexisting amenities area equals total neighborhood area):

$$\hat{n}(x,y)\hat{h}_i(x,y) + a(x,y)l(x,y) + z(x,y)l(x,y) = l(x,y). \tag{7}$$

6. Total population satisfies:

$$\sum_{(x,y)\in\Theta} \hat{n}(x,y) = N \text{ in an open city,} \tag{8a}$$

$$\sum_{(x,y)\in\Theta} \hat{n}(x,y) = \bar{N} \text{ in a closed city} \quad \text{and} \quad \hat{n}(x,y)\geq 0 \quad \text{for all neighborhoods } (x,y)\in\Theta. \tag{8b}$$

3.2. Equilibrium results

The market equilibrium conditions along with the assumptions of the model can be used to demonstrate several general results of the effect of open space on land rental prices and development patterns. For the closed-city model, we also need an additional condition in order to prove the first proposition.

Condition 1. For any two neighborhoods (x,y) and (s,r) such that $x\neq s$ or $y\neq r$, where $u(c_i(x,y),h_i(x,y),A(x,y)) = u(c_i(s,r),h_i(s,r),A(s,r))$, then $(\partial u(c_i(x,y),h_i(x,y),A(x,y))/\partial A(x,y))(\partial A(x,y)/\partial a(x,y)) > (\partial u(c_i(s,r),h_i(s,r),A(s,r))/\partial A(s,r))(\partial A(s,r)/\partial a(x,y))$.

Condition 1 will be satisfied in most but not all circumstances. Because the "own-neighborhood" amenity effect is larger than the "cross-neighborhood" amenity effect for an increase in open space: $\partial A(x,y)/\partial a(x,y) > \partial A(s,r)/\partial a(x,y)$, Condition 1 will hold unless the marginal utility of amenities is much higher in other neighborhoods. Condition 1 may be violated if there is decreasing marginal utility of amenities and other neighborhoods begin with far fewer amenities.

Proposition 1. *Assuming the consumption good and housing are normal goods, an increase in open space in neighborhood (x,y) will increase the after-tax rental price of land in the neighborhood, $p^\tau(x,y)$, and increase household density in the developed area of the neighborhood, $n(x,y)/(l(x,y)(1-a(x,y)-z(x,y)))$, for the open-city model and, assuming Condition 1, for the closed-city model.*

The proofs of this proposition and all following propositions are given in Appendix A.

An increase in open space in a neighborhood increases the amenity value of that neighborhood, leading to an increase in utility of living in the neighborhood other things constant. In the open-city model, the increase in utility will attract people to the neighborhood, bidding up price and increasing housing density. In the closed-city model, an increase in open space will cause a greater increase in utility in that neighborhood than in others (Condition 1) thereby attracting people to the neighborhood thereby bidding up price and increasing housing density.

Proposition 1 shows that an increase in open space increases the price that residents in that neighborhood pay for land, $p^{\tau}(x, y)$, and decreases the amount of housing consumed per household so that housing density in the developed area increases (assuming Condition 1 for a closed-city model). However, because increasing open space increases taxes and takes some land out of development, the overall effect of an increase in open space on neighborhood population and the total value of developed land in the neighborhood is ambiguous. Often, it is these type of aggregate effects that are of greatest interest. For example, municipal leaders might be interested in knowing whether the tax base would increase or decrease with an increase in open space. If the pull of the open space amenity on demand is strong enough, both population and the value of developed land within the neighborhood will increase with an increase in open space. However, if the supply side push from lowering the amount of developable land in the neighborhood is stronger, neighborhood population and the total value of developed land will fall with an increase in open space. Similarly, the pre-tax rental price that landowners receive may increase or decrease with an increase in open space.

In the open-city model, we are able to derive several analytical results using the general framework, including the effect of increased open space in one neighborhood on prices and housing density in other neighborhoods.

Proposition 2. *Assuming the consumption good and housing are normal goods, an increase in open space in neighborhood (x, y) will result in an increase in the after-tax rental price of land in other neighborhoods, $p^{\tau}(s, r)$, and an increase in household density in the developed area of other neighborhoods, $n(s, r)/(l(s, r)(1 - a(s, r) - z(s, r)))$, for the open-city model when there are positive spillover effects. There will be no effect on after-tax rental price of land or household density in the open-city model when spillover effects are zero.*

In the case of a closed city with a fixed population, an increase in open space in one neighborhood can either increase or decrease the after-tax rental price of land and density in other neighborhoods. Price and density in other neighborhoods tends to increase because open space provision reduces developable land, which increases the pressure on remaining developable land in all neighborhoods. This effect in other neighborhoods is reinforced when there are strong amenity spillovers. On the other hand, if there are strong local amenity effects, this will tend to create a pull toward the neighborhood with open space, thereby reducing demand for other neighborhoods, which may result in lower prices and lower density.

Proposition 3. *In an open-city model, how open space is paid for, whether each neighborhood pays for its own open space or there is a city-wide property tax, affects pre-tax equilibrium rental prices of land, $p(x, y)$ but does not affect after-tax rental prices, $p^{\tau}(x, y)$, or decisions by households in equilibrium, $c_i(x, y), h_i(x, y), n(x, y)$, for any neighborhood that remains in the city.*

As long as the best use of the land in neighborhood (x, y) remains urban use rather than agriculture, the landlord will absorb the increased property tax leaving the post-tax rental price of land constant. Post-tax land rental price, $p^{\tau}(x, y)$, is fixed because the mobility of residents forces utility levels to $\bar{u}$. Pre-tax land rental price, $p(x, y)$, is pure rents to landowners. As long as $p(x, y) \geq p_{\mathrm{g}}(x, y)$, landowners will continue to rent to households (and the city government for open space). In this case, a property tax has no distortionary impacts. This case provides an example where the land tax promoted by George (1984) is the preferred form of taxation as all incidence falls on landowners who supply land inelastically, resulting only in redistributive not efficiency consequences from increased taxation. Only when $p(x, y) < p_{\mathrm{g}}(x, y)$ will landowners remove land from urban use and instead rent land to farmers. At this point increased property taxes are no longer neutral but will result in a smaller city with fewer neighborhoods and lower population.

3.3. Analytic solution

To make further progress, it is necessary to specify a functional form. With specific functional forms it is possible to obtain a closed-form analytic solution for equilibrium. Here we assume that the utility function is Cobb–Douglas:

$$u(c_i(x, y), h_i(x, y), A(x, y)) = \alpha \ln (c_i(x, y)) + \beta \ln (h_i(x, y)) + \gamma \ln A(x, y). \tag{9}$$

For each $(x, y) \in \Theta$ a household living in the neighborhood chooses consumption good, $c_i(x, y)$, and housing, $h_i(x, y)$, to maximize its utility function (Eq. (9)) subject to budget constraint (Eq. (4b)) and non-negativity conditions (4c). The Lagrangian for this utility maximization problem is

$$\begin{aligned} \mathcal{L} = {} & \alpha \ln c_i(x, y) + \beta \ln h_i(x, y) + \gamma \ln A(x, y) \\ & + \lambda(v - c_i(x, y) - p^{\tau}(x, y) h_i(x, y) - f[d_{\mathrm{C}}(x, y)]). \end{aligned}$$

The objective function is strictly increasing in both c and h and the $\lim_{h \to 0} u_c = \infty$ and $\lim_{n \to 0} u_h = \infty$. The objective function is also concave and subject to linear constraints. Therefore, there exists a unique interior solution to the household's maximization problem. The following first-order conditions are necessary and sufficient for this solution:

$$\frac{\partial \mathcal{L}}{\partial c_i(x, y)} = \frac{\alpha}{c_i(x, y)} - \lambda = 0, \qquad \frac{\partial \mathcal{L}}{\partial h_i(x, y)} = \frac{\beta}{h_i(x, y)} - \lambda p^{\tau}(x, y) = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = v - c_i(x, y) - p^{\tau}(x, y) h_i(x, y) - f[d_{\mathrm{C}}(x, y)] = 0.$$

The first-order conditions are solved for the household demand functions for the consumption good and housing:

$$c_i(x, y) = \frac{\alpha}{\alpha + \beta} (v - f[d_{\mathrm{C}}(x, y)]), \tag{10a}$$

$$h_i(x, y) = \frac{\beta}{\alpha + \beta} \left(\frac{v - f[d_{\mathrm{C}}(x, y)]}{p^{\tau}(x, y)} \right). \tag{10b}$$

3.3.1. Open city

For an open city, we use the household demand (Eqs. (10a) and (10b)), the total land area constraints (Eq. (7)), the no arbitrage condition across locations (5a), to find the equilibrium allocation, $\{\{\hat{c}_i(x,y),\hat{h}_i(x,y)\}_{i=1}^{\hat{n}(x,y)},\hat{n}(x,y)\}_{(x,y)\in\Theta}$, and prices, $\{\hat{p}^{\tau}(x,y)\}_{(x,y)\in\Theta}$ (see Appendix B for the full derivation of the solution):

$$\hat{c}_i(x,y)=\frac{\alpha}{\alpha+\beta}(v-f[d_{\mathrm{C}}(x,y)]), \tag{11a}$$

$$\hat{h}_i(x,y)=\left(\frac{\alpha+\beta}{\alpha}\right)^{\alpha/\beta}\frac{\mathrm{e}^{\bar{u}/\beta}}{(v-f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}, \tag{11b}$$

$$\hat{p}^{\tau}(x,y)=\max\left\{\frac{\alpha^{\alpha/\beta}\beta}{(\alpha+\beta)^{(\alpha+\beta)/\beta}\mathrm{e}^{\bar{u}/\beta}}(v-f[d_{\mathrm{C}}(x,y)])^{(\alpha+\beta)/\beta}A(x,y)^{\gamma/\beta},\,p_{\mathrm{g}}(x,y)\right\}, \tag{11c}$$

$$\hat{n}(x,y)=\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/\beta}\mathrm{e}^{-\bar{u}/\beta}[1-a(x,y)-z(x,y)]l(x,y)(v-f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}. \tag{11d}$$

3.3.2. Closed city

For a closed city, we use the household demand (Eqs. (10a) and (10b)), the total land area constraints (Eq. (7)), the no arbitrage condition across locations (5b), and the fact that total population is fixed at $\bar{N}$ (Eq. (8b)), to find the equilibrium allocation, $\{\{\tilde{c}_i(x,y),\tilde{h}_i(x,y)\}_{i=1}^{\tilde{n}(x,y)},\tilde{n}(x,y)\}_{(x,y)\in\Theta}$, and prices, $\{\tilde{p}^{\tau}(x,y)\}_{(x,y)\in\Theta}$ (see Appendix C for the full derivation of the solution):

$$\tilde{c}_i(x,y)=\frac{\alpha}{\alpha+\beta}(v-f[d_{\mathrm{C}}(x,y)]), \tag{12a}$$

$$\tilde{h}_i(x,y)=\frac{\sum_{(s,r)\in\Theta}\left\{(1-a(x,y)-z(x,y))l(x,y)(v-f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\right\}}{\bar{N}(v-f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}, \tag{12b}$$

$$\tilde{p}^{\tau}(x,y)=\max\left\{\frac{\beta\bar{N}(v-f[d_{\mathrm{C}}(x,y)])^{(\alpha+\beta)/\beta}A(x,y)^{\gamma/\beta}}{(\alpha+\beta)\sum_{(s,r)\in\Theta}\{(1-a(x,y)-z(x,y))l(x,y)(v-f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\}},\,p_{\mathrm{g}}\right\}, \tag{12c}$$

$$\tilde{n}(x,y)=\frac{\bar{N}(1-a(x,y)-z(x,y))l(x,y)(v-f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}{\sum_{(s,r)\in\Theta}\{(1-a(x,y)-z(x,y))l(x,y)(v-f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\}}. \tag{12d}$$

With this solution it is straight-forward to demonstrate the effect of adding open space in a given neighborhood on the pattern of development and land prices in the urban area. Another advantage of this model is that it is easy to add a variety of amenities (lakes, wetlands, hills) and disamenities (waste sites, smokestacks) in addition to open space amenities.

Given the open space distribution $\{a(x,y)\}_{(x,y)\in\Theta}$, the preexisting amenities $\{z(x,y)\}_{(x,y)\in\Theta}$, and the business centers $\{b(x_j,y_j)\}_{j=1}^{J}$, we use the housing price $p^{\tau}(x,y)$ and demand $h_i(x,y)$ equations to solve for the fringe neighborhoods using condition (6) from the definition of the equilibrium.

4. Optimal provision of open space

In the analysis of equilibrium in the previous section, the allocation of open space across neighborhoods is given. Here we tackle the problem of optimal provision of open space when there are amenity spillover effects among neighborhoods. This problem has been analyzed previously for a restricted single-dimensional example with five neighborhoods in Yang and Fujita (1983). As the algebra gets complicated with amenity spillover, we restrict our analysis to the case of the open-city model. In solving for the optimal allocation, it is easier to work with the bid-rent function rather than the direct demand functions as has been done above. Assuming that the utility function is concave in $c_i(x, y)$, $h_i(x, y)$, and $A(x, y)$, the household utility maximization problem can be restated as a bid-rent function maximization problem. That is, in order for a household to maintain a utility level $\bar{u}$, what is the maximum price that this household is willing to pay to reside in some neighborhood (x, y)? This problem is as follows:

$$\max_{c_i(x,y)} \frac{v - c_i(x, y) - f[d_{\mathrm{C}}(x, y)]}{h_i(x, y)}$$

such that:

$$u[c_i(x, y), h_i(x, y), A(x, y)] \geq \bar{u}, \quad c_i(x, y) \geq 0.$$

The solution of this problem is the bid-rent function:

$$\frac{v - c_i(h_i(x, y), A(x, y), \bar{u}) - f[d_{\mathrm{C}}(x, y)]}{h_i(x, y)},$$

where $c_i(h_i(x, y), A(x, y), \bar{u})$ is the inverse function of $u[c_i(x, y), h_i(x, y), A(x, y)] = \bar{u}$. Given this bid function, the optimal allocation, $\{\{h_i^*(x, y)\}_{i=1}^{n^*(x,y)}, n^*(x, y), a^*(x, y)\}_{(x,y)\in\Theta}$, solves the following problem:

$$\max_{\{h_i(x,y),n(x,y),a(x,y)\}_{(x,y)\in\Theta}} \sum_{(x,y)\in\Theta} \left\{ \left(\frac{v - c_i(h_i(x, y), A(x, y), \bar{u}) - f[d_{\mathrm{C}}(x, y)]}{h_i(x, y)} - p_{\mathrm{g}}(x, y) \right) \times n(x, y)h_i(x, y) - a(x, y)l(x, y)\, p_{\mathrm{g}}(x, y) \right\} \tag{13a}$$

subject to

$$\begin{aligned} &n(x, y)h_i(x, y) + a(x, y)l(x, y) + z(x, y)l(x, y) \leq l(x, y) \quad \text{for all } (x, y) \in \Theta \\ &n(x, y) \geq 0, \quad h_i(x, y) \geq 0, \quad 1 \geq a(x, y) \geq 0 \quad \text{for all } (x, y) \in \Theta, \end{aligned} \tag{13b}$$

where the objective function is the bid-rent function net of the opportunity cost of an alternative land use pattern. To simplify the optimization problem, first observe that the total land constraint will hold with equality because household's preferences are strictly increasing in housing, so we can substitute $(1 - a(x, y) - z(x, y))l(x, y)$ from Eq. (13b) for $n(x, y)h_i(x, y)$ in the objective function (13a). Second, observe that after the substitution the optimal housing allocation as a function of amenities, distance, and utility level, can be obtained by solving

$$\max_{h_i(x,y)} \frac{v - c_i(h_i(x, y), A(x, y), \bar{u}) - f[d_{\mathrm{C}}(x, y)]}{h_i(x, y)}.$$

If we use the utility function specification given in Eq. (9), then the solution of this maximization problem in housing is

$$h_i(x,y)=\left(\frac{\alpha+\beta}{\alpha}\right)^{\alpha/\beta}\frac{e^{\bar{u}/\beta}}{(v-f[d_C(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}.$$

With these two observations, the utility function, and rearranging the terms, the optimization problem can be restated as follows:

$$\max_{\{a(x,y)\}_{(x,y)\in\Theta}}\sum_{(x,y)\in\Theta}\left(\begin{array}{r}\dfrac{\alpha^{\alpha/\beta}\beta}{(\alpha+\beta)^{(\alpha+\beta)/\beta}e^{\bar{u}/\beta}}(v-f[d_C(x,y)])^{(\alpha+\beta)/\beta}A(x,y)^{\gamma/\beta}(1-a(x,y)\\ -z(x,y))l(x,y)-p_g(x,y)l(x,y)\end{array}\right)$$

$$\text{subject to}: a(x,y)\in[0,1]\quad \text{for all}\,(x,y)\in\Theta.$$

The solution is the optimal allocation of open space $\{a^*(x,y)\}_{(x,y)\in\Theta}$. Using this optimal allocation of open space we solve for the optimal allocations of housing $\{\{h_i^*(x,y)\}_{i=1}^{n^*(x,y)}\}_{(x,y)\in\Theta}$, and density $\{n^*(x,y)\}_{(x,y)\in\Theta}$.

Similar to the results in Yang and Fujita (1983), given the optimal allocation of open space, $\{a^*(x,y)\}_{(x,y)\in\Theta}$, and the same level of utility and preexisting amenities, the market equilibrium allocation $\{\{\hat{c}_i(x,y),\hat{h}_i(x,y)\}_{i=1}^{\hat{n}(x,y)},\hat{n}(x,y)\}_{(x,y)\in\Theta}$, and the optimal allocation, $\{\{c_i^*(x,y),h_i^*(x,y)\}_{i=1}^{n^*(x,y)},n^*(x,y)\}_{(x,y)\in\Theta}$, are identical.

Unlike the equilibrium solution shown above, the optimal solution with amenity spillovers does not have a closed-form analytical solution. In the next section, we solve for the optimal spatial allocation of open space using numerical simulation.

5. Simulation results

In this section, we determine the optimal spatial pattern of open space, equilibrium population and land rents for an open-city model using numerical simulation for two examples. The first example is a symmetric city with a single central business district and no preexisting amenities. This example is relatively transparent and we use it to highlight the effects of changes in amenity spillovers and transportation costs on the optimal spatial pattern of open space, population, and equilibrium land rents. The second example is based on data from the Twin Cities Metropolitan Area (Minneapolis–St. Paul, MN, USA). The Twin Cities have two downtowns, one in Minneapolis and another in St. Paul, and a large number of existing lakes and parks. This example is used to explore potentially asymmetric solutions as well as to demonstrate how such a model might be parameterized and applied to a metropolitan area with multiple employment centers and a large variety of preexisting features such as commercial development, parks, and natural landscapes.

5.1. Symmetric example

In this first example, we optimize open space allocation for a symmetric city with a central business district (CBD) located at the origin $(0,0)$ on a 25×25 grid of square neighborhoods. Each neighborhood has unit area and has its center represented by its (x,y) coordinates. We use a linear transportation cost function: $f[d_C(x,y)]=\sigma d_C(x,y)$. We experimented with an alternative specification of the transportation cost function, specifically an exponential cost function

$f[d_C(x,y)] = \exp[\sigma d_C(x,y)]$ (Lucas and Rossi-Hansberg, 2002). We obtained qualitatively similar results with both transportation cost functions and so only report results using the linear cost function here. In general, a variety of transportation cost forms can be used because the model is quite flexible. We specify the amenity function (Eq. (3)) for neighborhood $(x,y), A(x,y)$, to be

$$A(x,y) = \sum_{(s,r)\in\Theta} [\delta_a a(s,r) + \delta_z z(s,r)] l(s,r) \exp[-\phi d(x-s, y-r)] + \sum_{(s,r)\in\Omega} [\delta_z z(s,r) + \delta_g g(s,r)] l(s,r) \exp[-\phi d(x-s, y-r)], \tag{14}$$

where $a(s,r)l(s,r)$ is the area of open space in neighborhood (s,r), $z(s,r)l(s,r)$ the area of preexisting amenity in neighborhood (s,r), $g(s,r)l(s,r)$ the area of agricultural land in neighborhood (s,r), and δ_a, δ_z, δ_g, are the amenity value weights for open space, preexisting amenities, and agricultural land, respectively, with $\delta_a > 0$. The values of δ_z and δ_g may be either positive (amenity), zero, or negative (disamenity). ϕ is the parameter that measures the effect of distance to an amenity, and $d(x-s, y-r) = \sqrt{(x-s)^2 + (y-r)^2}$ is the distance between neighborhoods (x,y) and (s,r).

The parameter values used in the simulation are given in Table 1. We vary the amenity spillover effect and the transportation cost across different simulations. Because the parameter values for a given simulation are homogeneous across neighborhoods, we assume that the optimal spatial pattern of open space, population, and land rents is symmetric around the CBD. For each set of parameter values, we determine the optimal pattern of open space for cities of increasing radius until the equilibrium land rent on the perimeter is equal to the price of agricultural land. The optimization problem is formulated and solved using GAMS and the nonlinear solver CONOPT. The algorithm found the same identical solution for a given problem even when we initiated the algorithm at different starting points adding to our confidence that we have indeed found a unique optimal solution.

We begin by showing the effect of amenity spillover from open space in one neighborhood to other neighborhoods on the optimal provision of open space and the resulting equilibrium population and land rent in the city. When the distance weight for the amenity spillover effect is large ($\phi = 5.0$), there is virtually no amenity spillover from one neighborhood to another so that open space is a local public good. Assuming no amenity spillover and a Cobb–Douglas utility function, it is optimal for all neighborhoods to contain the same amount of open space (Yang and Fujita, 1983, Theorem 5). The optimal amount of open space is equal to $\gamma/\beta + \gamma = 0.4$, where γ is the coefficient on amenities and β is the coefficient on housing in the Cobb–Douglas utility function. The results for this case are shown in Fig. 1. The city is relatively small in size (radius = 6 neighborhoods) and population (34 households). Neighborhoods closer to the CBD have slightly more households, higher density, and higher land prices because of the lower cost of commuting.

When the distance weight for the amenity spillover is small ($\phi = 0.1$), there is an amenity spillover effect. Open space in one neighborhood contributes to the utility of people in surrounding neighborhoods and throughout the city. Because people benefit from open space throughout the city, the city is larger in size (radius = 13 neighborhoods) and population (3373 households). The results with amenity spillovers are shown in Fig. 2. The optimal provision of open space is a greenbelt five neighborhoods wide on the city's perimeter with no open space located in central neighborhoods. This pattern of open space maintains amenity value for all neighborhoods but reduces transportation costs by concentrating people near the CBD. Because

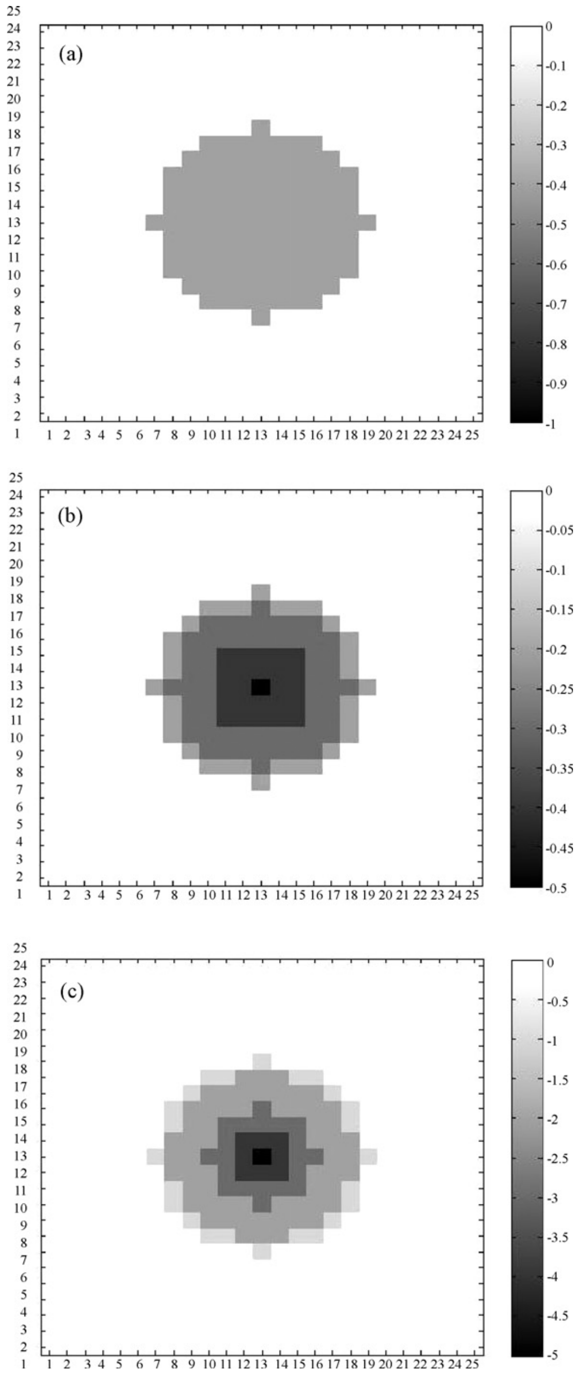

Fig. 1. Optimal levels of open space (a), households (b), and land price (c) for a symmetric city with no amenity spillover effect ($\phi = 5.0$) and high transportation cost ($\sigma = 1.0$).

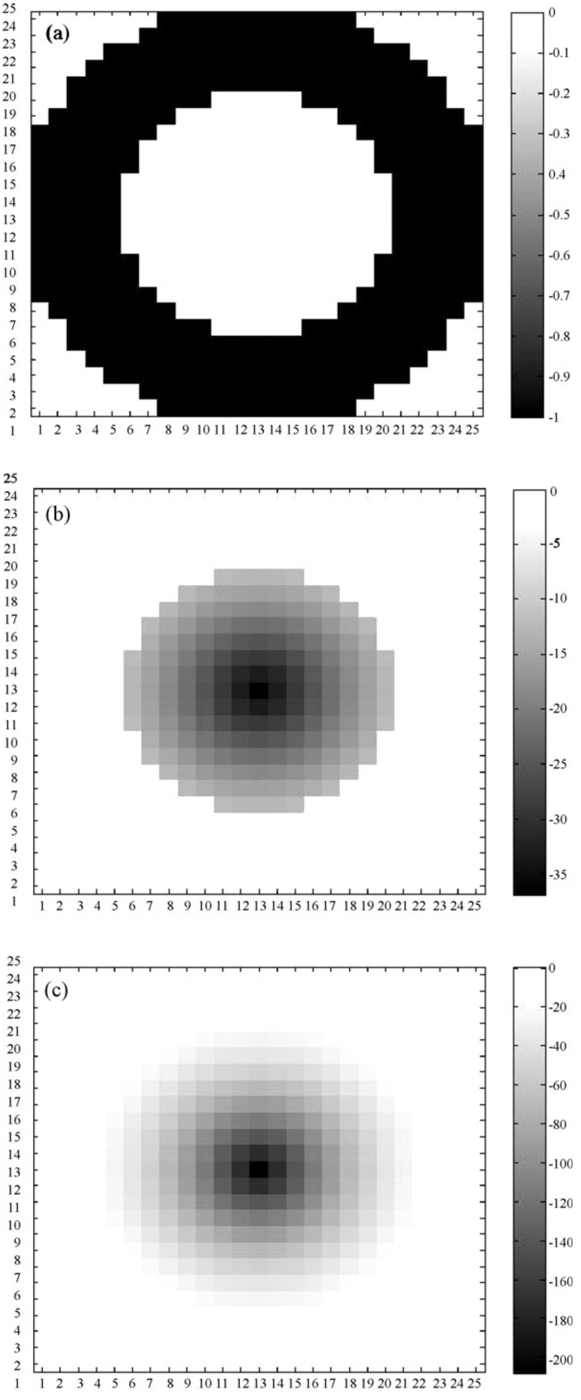

Fig. 2. Optimal levels of open space (a), households (b), and land price (c) for a symmetric city with amenity spillover effect ($\phi = 0.1$) and high transportation cost ($\sigma = 1.0$).

the only major difference in locations is related to commuting costs, land prices are very high near the CBD and decrease rapidly toward the city's perimeter.

With no amenity spillover effect ($\phi = 5.0$), the optimal provision of open space is the same across all neighborhoods and is not affected by changes in transportation cost. With an amenity spillover effect ($\phi = 0.1$), transportation cost has a big impact on optimal provision of open space and on city size. When transportation cost per unit distance (σ) is reduced from 1.0 to 0.01, the radius of the city increases by two orders of magnitude to about 1300 neighborhoods for a value of agricultural land of $p_g = 1.0$. With over 5 million neighborhoods, the optimization problem exceeded the limits of our software. Therefore, we limited the city to a radius of 13 neighborhoods and computed optimal open space and population within this city size constraint.

With low transportation cost and a limit on the radius of the city, open space is located throughout the city except in perimeter neighborhoods and a small set of neighborhoods in the center of the city (Fig. 3). On the border of the city, there is a densely populated belt without open space and next to this, inside the city, there is a ring of neighborhoods that have a higher proportion of open space than neighborhoods closer to the CBD giving the impression of a greenbelt. Both the populated belt and the greenbelt are small (i.e., only a few neighborhoods in width). Further inside the city, the distribution of open space is relatively homogeneous except for a few neighborhoods surrounding the CBD. Total population (8299) is more than twice the number of households in a city with higher transportation cost, and more than half the population is concentrated in the ring three neighborhoods wide on the city's perimeter.

This pattern of housing and open space results from the combined effects of low transportation cost, amenity spillover, and limited city size. Within a city of predefined size, transportation costs are almost negligible: moving one neighborhood further outside the city costs only 0.07% of income. On the other hand, providing open space on the perimeter of the city carries considerable opportunity cost because almost half of the spillover benefits are wasted. Therefore, the optimal pattern is to provide more open space in a greenbelt close to the city's border and to move people to the neighborhoods outside this greenbelt on the city's perimeter. Further inside the city, the distribution of housing and open space is relatively homogeneous because low transportation cost blunts the advantage of living close to the CBD.

We conducted sensitivity analysis with regard to the simulated size of the city. As the radius of the city is increased, both belts on the perimeter of the city (the outer ring without open space and the greenbelt) move outward but remain of similar width. Although we did not compute the optimal provision of open space in a city with endogenous boundaries where $p_g = 1.0$, we speculate that it would have the same pattern as the city shown in Fig. 3.

5.2. Twin Cities simulation results

In this section, we demonstrate the flexibility of the discrete-space open-city optimization model by applying it to a case with two business centers and asymmetric distributions of existing amenities and development using data for the Twin Cities Metropolitan Region, Minnesota, USA. The study area is 14,300 km^2 and includes the cities of Minneapolis and St. Paul and surrounding area (Fig. 4). We divided the study area into a 13×11 grid of 100 km^2 square neighborhoods, each centered around its (x, y) coordinates with the neighborhood in the lower left corner labeled (1, 1). Two business centers representing downtown Minneapolis and downtown St. Paul are located in neighborhoods (6, 6) and (8, 6), respectively. For each neighborhood, we calculated the proportions of the neighborhood in existing commercial development, parks and water (Fig. 5) using the 2000 Generalized Land Use data set for the

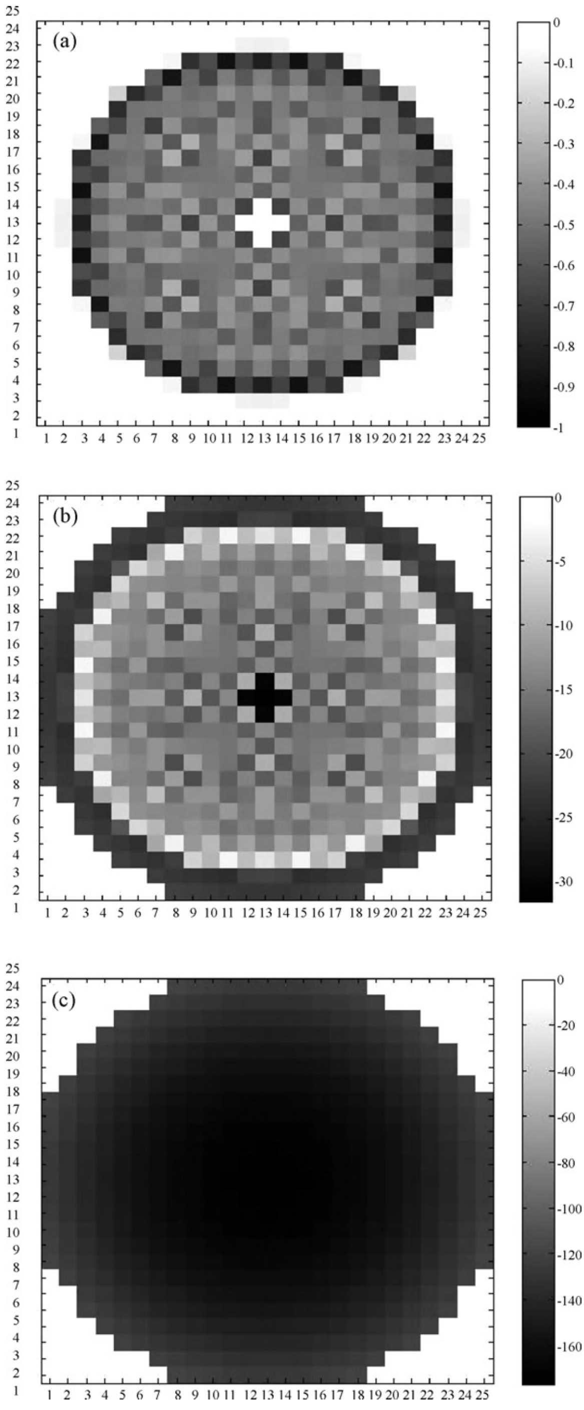

Fig. 3. Optimal levels of open space (a), households (b), and land price (c) for a symmetric city with amenity spillover effect ($\phi = 0.1$) and low transportation cost ($\sigma = 0.01$).

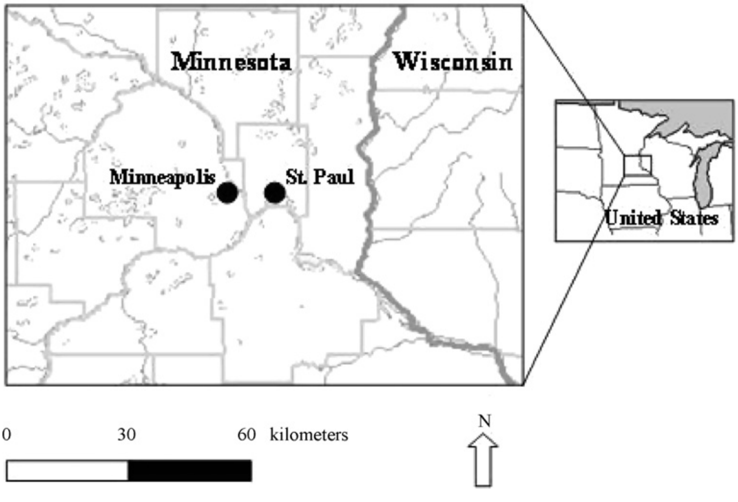

Fig. 4. The Twin Cities study area.

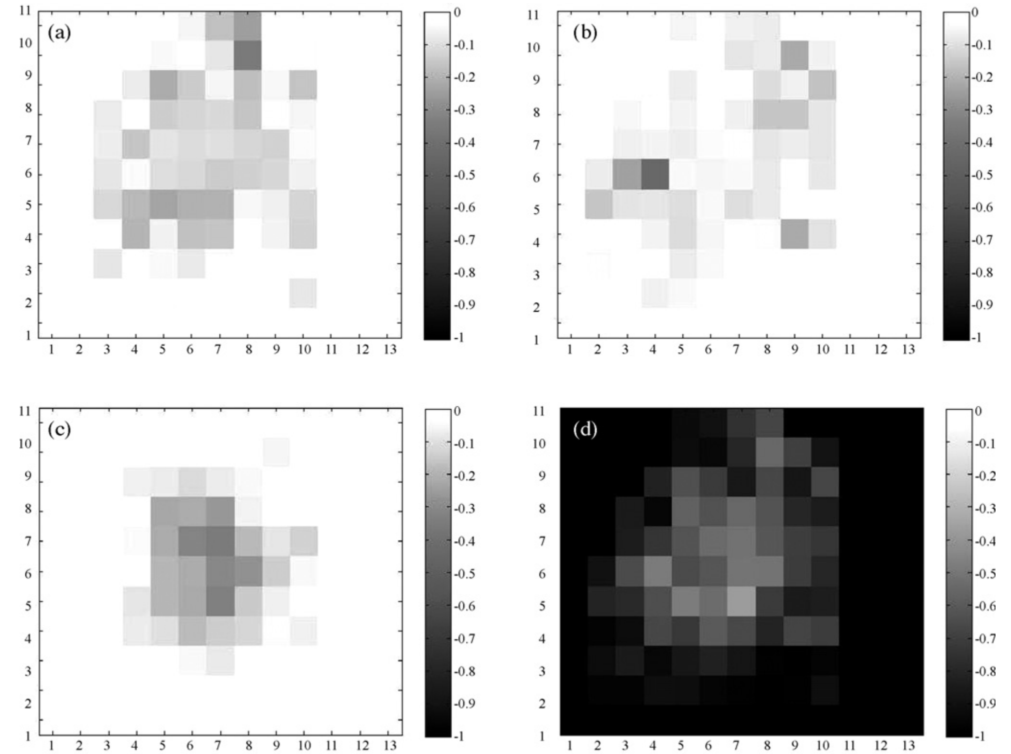

Fig. 5. Proportions of 100 km^2 neighborhoods currently covered by parks (a), lakes (b), commercial development (c), and ''undeveloped'' land (total land minus parks, lakes, and commercial development) (d) in the Twin Cities study area.

Table 1
Parameter values used in the symmetric simulations

Parameter description	Parameter value
Consumption good share (α)	0.5
Housing share (β)	0.3
Amenities share (γ)	0.2
Income (v)	15
Utility level in an open city ($\bar{u}$)	1.0
Price of agricultural land (p_g)	1.0
Transportation cost (σ) (high and low cost)	1.0 and 0.01
Distance effect on amenity (ϕ) (without and with spillover)	5.0 and 0.1
Amenity value for a unit of open space (δ_a)	1
Amenity value for a unit of preexisting features (δ_z)	1
Amenity value for a unit of agricultural land (δ_g)	0

seven-county Twin Cites Metropolitan Area (Metropolitan Council, 2007). Parks and water features (lakes, rivers) are treated as amenities. Some of the eastern neighborhoods in our grid spill into an area outside of the Twin Cities Metropolitan Area data. For the purposes of the simulation we assumed that these neighborhoods are undeveloped. Proportions of commercial development are greatest near the central business districts. Most neighborhoods have less than 10% of their area in amenities. The neighborhoods with more amenities are located around Lake Minnetonka (4, 6), Forest Lake (9, 10), and Spring Lake along the Mississippi River (9, 4). We assumed that existing commercial development and amenities were fixed at their current levels and defined 'undeveloped' land as total land minus the area of existing parks, lakes, and commercial development. We then determined the optimal amount of open space, population level, and equilibrium land rent for developable land in each neighborhood.

The parameter values used in the open-city optimization model are given in Table 2. We assumed the same share parameters (α, β, γ) in the utility function as in the symmetric example. We used annual household income of \$40,915 for Minneapolis–St. Paul in 2004 (Bureau of Economic Analysis, 2007). We set the reservation utility level so that the Twin Cities would grow from its current size (continuing a pattern of growth in recent decades). We assumed the price of agricultural land to be \$4200 per ha (\$1700 per acre). Annual transportation cost for commuting

Table 2
Parameter values used in the twin cities simulation

Parameter description	Parameter value
Consumption good share (α)	0.5
Housing share (β)	0.3
Amenities share (γ)	0.2
Income (v) \$1000 per year	40.9
Utility level in an open city ($\bar{u}$)	3.0
Price of agricultural land (p_g) \$1000 per ha	4.2
Transportation cost (σ) \$1000 per km per year	0.331
Distance effect on amenity (ϕ)	0.1
Amenity value for a unit of open space (δ_a)	1
Amenity value for a unit of preexisting features (δ_z)	1
Amenity value for a unit of agricultural land (δ_g) (two levels)	0.0 and 0.5

to work was assumed to be \$331 per km. To compute annual transportation cost, we assumed a travel speed of 40 km/h, a wage of \$17.49 per h (hourly wage in Minneapolis–St. Paul in 2005; Minnesota Department of Employment and Economic Development, 2007), fuel and vehicle maintenance costs of \$0.20 per km, and 520 one-way commutes per year (5 round trips per week for 52 weeks). We assumed a positive amenity spillover effect ($\phi = 0.1$) so that open space in one neighborhood contributes to the utility of people in surrounding neighborhoods. We assumed that existing amenities entered the amenity function in the same fashion as open space. We performed the optimization for two different levels of the amenity parameter for agricultural land (0 and 0.5).

When the amenity parameter for agricultural land is zero, the optimal pattern of new open space is a greenbelt surrounding the Twin Cities Metropolitan Area (Fig. 6) 30–40 km from the central business districts. The parameter values in the Twin Cities example have relatively high transportation cost and a high degree of amenity spillover. The optimal pattern of open space for the Twin Cities is consistent with the results obtained for the symmetric case with amenity spillovers and high transportation cost. With amenity spillover and relatively high transportation cost, people choose to live close to the city centers with open space located on the perimeter. The greatest proportion of housing occurs in neighborhoods near the perimeter that currently have higher proportions of undeveloped land. Land prices decline symmetrically as distance from central business districts increases. Population is somewhat asymmetrically distributed,

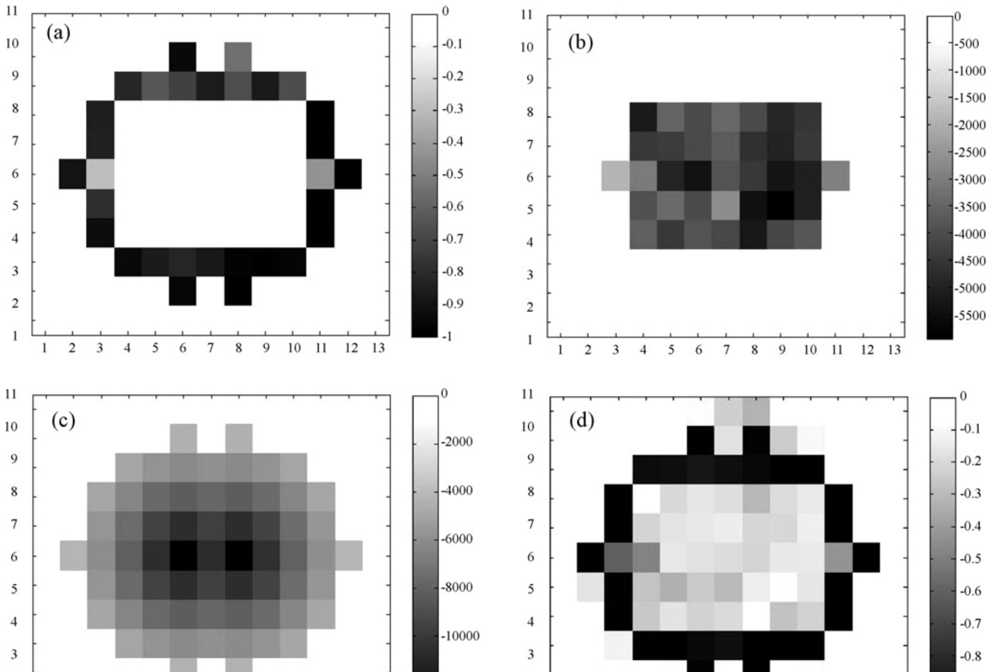

Fig. 6. Optimal levels of new open space (a), households (b) and land price (c) given existing levels of parks, lakes, and commercial development in the Twin Cities study area. Map (d) shows the proportions of neighborhoods covered by new open space plus existing parks and lakes. Undeveloped land outside the city boundary has no amenity value.

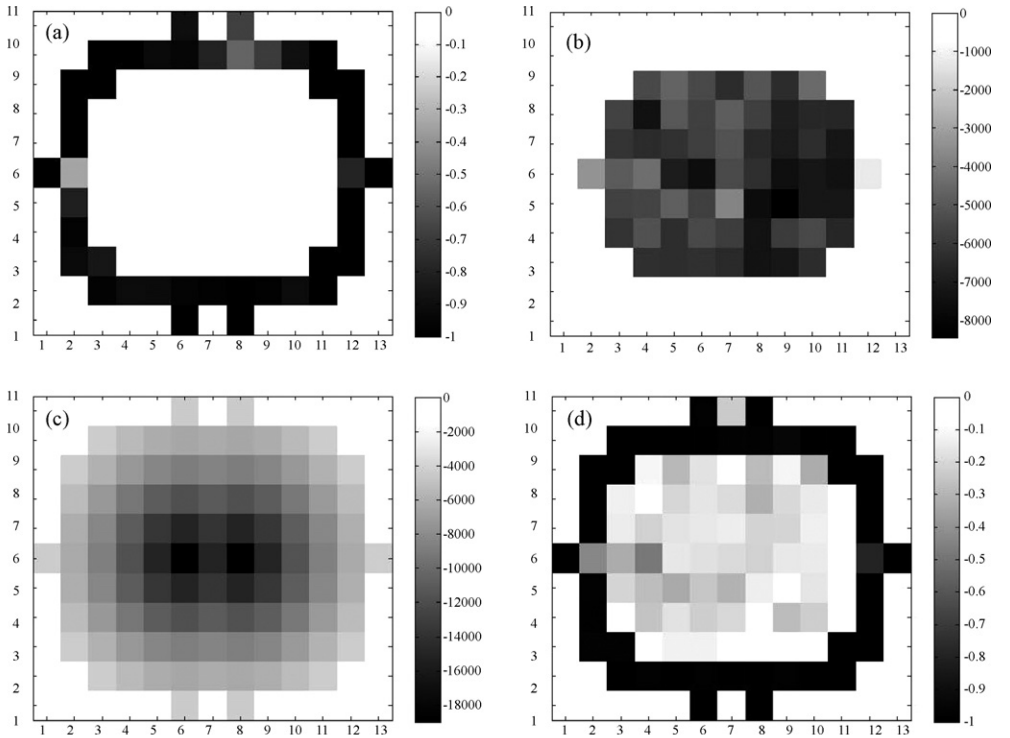

Fig. 7. Optimal levels of new open space (a), households (b) and land price (c) given existing levels of parks, lakes, and commercial development in the Twin Cities study area. Map (d) shows the proportions of neighborhoods covered by new open space plus existing parks and lakes. Undeveloped land outside the city boundary has amenity value.

reflecting the supply of developable land in a neighborhood once existing amenities and commercial development is subtracted.

Increasing the amenity parameter for agricultural land increases the size of the city but maintains the same optimal pattern of amenities (Fig. 7). New open space is located in a greenbelt surrounding the metropolitan area 40–50 km from the central business districts. The positive amenity associated with agricultural land makes the city more desirable, thereby attracting more people, increasing land rents, and increasing the size of the city.

6. Discussion

We analyzed the effects of open space and associated environmental amenities in a discrete-space urban model. The discrete-space model allowed us to determine how the size and location of open space together with the degree of amenity spillover across neighborhoods affected the equilibrium pattern of housing density and land price. Our discrete-space model contrasts with most urban economics models in which environmental amenities do not take up space and are characterized by their distance to the central business district (notable exceptions being Wu and Plantinga, 2003; Wu, 2006). We find that provision of open space within a neighborhood increases after-tax land values in the neighborhood and increases housing density in the developed portion of the neighborhood. This happens because open space provides local environmental amenities that make the neighborhood more desirable and reduces the area of developable land. However, the overall effect of an increase in open space on total number of households and total value of developed land in the neighborhood is ambiguous and depends upon whether the pull from open space amenities is stronger than the push from the reduction in developable land. An increase in open space in one neighborhood affects housing density and land prices in other neighborhoods in an open-city model with amenity spillovers. In this case, land price and housing density rise in other neighborhoods because they become more attractive places to live. In a closed-city model with amenity spillover effects, an increase in open space in one neighborhood may increase or decrease housing density and land price in other neighborhoods depending on whether the pull of the increased amenity in the neighborhood outweighs the push resulting from the reduced amount of developable land in the neighborhood.

We also formulated and solved the problem of determining the optimal size and location of open space across neighborhoods. Our formulation builds on Yang and Fujita (1983), who analyzed equilibrium and optimal provision of open space with a one-dimensional urban model. Going beyond Yang and Fujita (1983), we determined the optimal pattern of open space in a two-dimensional discrete-space model that allows amenity spillover effects across neighborhoods. As in Yang and Fujita (1983), we found that, when neighborhoods are of equal size, preferences are Cobb–Douglas, and there are no amenity spillovers, it is optimal to provide the same amount of open space in all neighborhoods. With amenity spillover effects, we found that the optimal city size and pattern of open space depend on transportation cost. With high transportation cost, it is optimal to provide more open space in neighborhoods on the edge of the city far from the central business district rather than the interior neighborhoods. Doing so reduces commuting costs to the central business district for the majority of city residents while still giving all neighborhoods the benefits of open space because of the spillover effects. With low transportation cost, more people move into the city increasing its size and population. Open space is located in interior neighborhoods and most people live in perimeter neighborhoods because cost of living far from the central business district is reduced and open space in perimeter districts has fewer spillover effects than open space in interior districts.

The strength of our discrete-space model is that it can incorporate realistic features such as multiple business districts, existing environmental amenities, and amenity values of agricultural land. In an application based on data for the Twin Cities of Minneapolis and St. Paul, USA, the optimal pattern of new open space when added to existing lakes and parks is a greenbelt on the city boundary about 30 km from the two central business districts. Assigning a positive amenity value to agricultural land outside the city boundary increased the size of the city but maintained the same optimal pattern of open space.

At present there is a large gap between the highly stylized general equilibrium spatial models of much of urban economics, and the largely empirical partial equilibrium models of the value of amenities. An important goal for research is to close this gap, and the discrete-space model developed here is promising because its analytical framework accounts for the size and location of new and existing amenities, multiple employment centers, and amenity spillover effects. In addition, the discrete-space model can be extended to include other important features such as income classes, transportation networks, and multiple political jurisdictions. In many metropolitan areas there are multiple political jurisdictions that each control land use decisions for some portion of the area. The discrete-space model could be used to solve for equilibrium in a game among multiple agents each of which makes decisions on taxes and provision of open space. Such an analysis could show the degree to which lack of coordination among jurisdictions leads to inefficiency and skewed patterns of development. Expanding the discrete-space urban model along these lines could yield further insights into the general equilibrium spatial effects of the provision of open space and other environmental amenities.

Acknowledgment

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Appendix A. Proofs of the propositions

A.1. Proof of Proposition 1

Proof. Let the initial proportion of open space in neighborhood (x, y) be $a^0(x, y)$ and suppose that this increases to $a^1(x, y)$. Because the own-neighborhood amenity effect of open space is positive, $\partial A(x, y)/\partial a(x, y) > 0$, environmental amenities in neighborhood (x, y) after the increase in open space, $A^1(x, y)$, will be greater than the initial level of environmental amenities $A^0(x, y)$: $A^1(x, y) > A^0(x, y)$.

(i) Open-city model: In equilibrium, the utility of households living in all neighborhoods equals $\bar{u}$. With the initial level of open space $u[c_i(x, y), h_i(x, y), A^0(x, y)] = \bar{u}$. Increasing open space in neighborhood (x, y) increases environmental amenities to $A^1(x, y)$, which increases utility above $\bar{u}$, holding other things constant. To reestablish equilibrium with $A^1(x, y)$, and given that income and the price of the consumption good are fixed, the post-tax rental price of land in the neighborhood, $p^\tau(x, y)$, must rise. Because housing is a normal good, with higher rental prices of land, households living in (x, y) will choose to consume less housing, so that $h_i^1(x, y) < h_i^0(x, y)$. A decline in amount of housing chosen by each household results in an increase in household density, completing the proof for the open-city model.

(ii) Closed-city model: The proof in a closed-city model is somewhat more involved. By assumption, housing is a normal good. Therefore, by the Slutsky equation, $h_i(x, y)$ is decreasing in its after-tax price, $p^\tau(x, y)$. To prove the proposition then, it is sufficient to show that $h_i(x, y)$, which is inversely related to housing density and price, must decline with an increase in open space in neighborhood (x, y). Suppose $c_i^0(x, y), h_i^0(x, y)$, $n^0(x, y)$ and prices $p^{\tau 0}(x, y)$ constitute an equilibrium given the initial amount of open space. If the proportion of open space in neighborhood (x, y) increases from $a^0(x, y)$ to $a^1(x, y)$, then because of the constraint on land area, $n(x, y)h_i(x, y) + a(x, y)l(x, y) + z(x, y)l(x, y) = l(x, y)$, we have that $n^1(x, y)h_i^1(x, y) < n^0(x, y)h_i^0(x, y)$, where $n^1(x, y), h_i^1(x, y)$ represent equilibrium values after the increase in open space. This implies that either the amount of housing consumed by each household in (x, y) declines with an increase in open space, $h_i^1(x, y) < h_i^0(x, y)$, and/or the total number of households who choose to live in that neighborhood declines, $n^1(x, y) < n^0(x, y)$. The proof proceeds by showing that $h_i^1(x, y) \geq h_i^0(x, y)$ results in a contradiction.

Suppose that $h_i^1(x, y) \geq h_i^0(x, y)$, so that $n^1(x, y) < n^0(x, y)$. From the constraint that total population is fixed, $\sum_{(x,y) \in \Theta} n(x, y) = \bar{N}$, we must have $n^1(s, r) > n^0(s, r)$ for at least one other neighborhood, (s, r) in the city. Since developable land in neighborhood (s, r) is constant, $n^1(s, r) > n^0(s, r)$ means that $h_i^1(s, r) < h_i^0(s, r)$. Initially, before the expansion of open space in neighborhood (x, y), the utility function for each household evaluated at the optimal choice was equal across all neighborhoods: $u[c_i^0(x, y), h_i^0(x, y), A^0(x, y)] = u[c_i^0(s, r), h_i^0(s, r), A^0(s, r)] = u$. By Condition 1, the marginal utility of an increase in open space in (x, y) is larger than in other neighborhoods: $(\partial u(c_i(x, y), h_i(x, y), A(x, y))/\partial A(x, y))(\partial A(x, y)/\partial a(x, y)) > (\partial u(c_i(s, r), h_i(s, r), A(s, r))/\partial A(s, r))(\partial A(s, r)/\partial a(s, r))$. Thus, both because of the change in utility from the change in amenities and from the change in utility from housing choice, we find that $u[c_i^1(x, y), h_i^1(x, y), A^1(x, y)] > u[c_i^1(s, r), h_i^1(s, r), A^1(s, r)]$, which cannot be an equilibrium.

Therefore, we conclude that $h_i^1(x, y) < h_i^0(x, y)$ with an increase in open space in neighborhood (x, y), which completes the proof. □

A.2. Proof of Proposition 2

Proof. In equilibrium, the utility of households living in all neighborhoods equals $\bar{u}$. With the initial level of open space this means that: $u[c_i^0(s, r), h_i^0(s, r), A^0(s, r)] = \bar{u}$. Holding other things fixed, increasing open space in neighborhood (x, y) when there are positive spillover effects increases environmental amenities to $A^1(s, r)$, which increases utility above $\bar{u}$. To reestablish equilibrium with $A^1(s, r)$, and given that income and the price of the consumption good are fixed, the post-tax rental price of land in the neighborhood, $p^\tau(s, r)$, must rise. Because housing is a normal good, with higher rental prices of land, households living in (s, r) will choose to consume less housing, i.e., $h_i^1(s, r) < h_i^0(s, r)$, which implies that household density increases. When there are no spillover effects, $A^1(s, r) = A^0(s, r)$, utility remains constant and there is no change in neighborhood (s, r). □

A.3. Proof of Proposition 3

Proof. A change in the tax rate in neighborhood (x, y), $\tau(x, y)$, does not affect the price of the consumption good, income, cost of commuting or the amenity level. In equilibrium in an open-city model, the utility of households living in all neighborhoods equals $\bar{u}$: $u[c_i(x, y), h_i(x, y), A(x, y)] = \bar{u}$. Because utility is fixed and taxes do not change other prices, income, amenities or commuting costs, a change in $\tau(x, y)$ cannot affect $p^\tau(x, y)$. Given that $p^\tau(x, y)$ is unchanged with a change in $\tau(x, y)$, decisions by households in equilibrium, $c_i(x, y), h_i(x, y), n(x, y)$, will not change with changes in $\tau(x, y)$ as long as the neighborhood remains in the city. □

Appendix B. Open-city equilibrium

From the condition of no arbitrage across locations (5a) and the Cobb–Douglas utility function (9) it follows that:

$$c_i(x, y)^\alpha h_i(x, y)^\beta A(x, y)^\gamma = \exp(\bar{u}). \tag{15}$$

Using the demand functions for the composite good (Eq. (10a)) and housing (Eq. (10b)) the utility condition (15) can be restated as

$$\left[\frac{\alpha}{\alpha+\beta}(v - f[d_C(x, y)])\right]^\alpha \left[\frac{\beta}{\alpha+\beta}\left(\frac{v - f[d_C(x, y)]}{p^\tau(x, y)}\right)\right]^\beta A(x, y)^\gamma = \exp(\bar{u}). \tag{16}$$

Solving this Eq. (16) for the after-tax housing price yields:

$$p^{\tau}(x,y) = \max\left\{ \frac{\alpha^{\alpha/\beta}\beta}{(\alpha+\beta)^{(\alpha+\beta)/\beta}\,\mathrm{e}^{-\bar{u}/\beta}}(v - f[d_{\mathrm{C}}(x,y)])^{(\alpha+\beta)/\beta}A(x,y)^{\gamma/\beta},\, p_{\mathrm{g}}(x,y)\right\}.$$

Plugging the above housing price equation into the housing demand function (10b) we solve for equilibrium housing:

$$h_i(x,y) = \left(\frac{\alpha+\beta}{\alpha}\right)^{\alpha/\beta} \frac{\mathrm{e}^{\bar{u}/\beta}}{(v - f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}.$$

Plugging in equilibrium housing in the land availability constraint (7) we solve for the number of households in each neighborhood:

$$n(x,y) = \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/\beta} \mathrm{e}^{-\bar{u}/\beta}[1 - a(x,y) - z(x,y)]l(x,y)(y - f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}.$$

The allocation of the composite good is already given in the demand equation (10a) as

$$c_i(x,y) = \frac{\alpha}{\alpha+\beta}(v - f[d_{\mathrm{C}}(x,y)]).$$

Given the open space distribution $\{a(x,y)\}_{(x,y)\in\Theta}$ use the housing price $p^{\tau}(x,y)$ and demand $h_i(x,y)$ equations to solve for the fringe neighborhoods using the condition (6) from the definition of the equilibrium.

Appendix C. Closed-city equilibrium

By rearranging the neighborhood land supply constraint (7) we have

$$n(x,y) = \frac{(1 - a(x,y) - z(x,y))l(x,y)}{h_i(x,y)}. \tag{17}$$

After plugging in for the household demand function (10b), Eq. (17) becomes

$$n(x,y) = \left(\frac{\alpha+\beta}{\beta}\right)\frac{(1 - a(x,y) - z(x,y))l(x,y)}{y - f[d_{\mathrm{C}}(x,y)]}\, p^{\tau}(x,y). \tag{18}$$

Using Eq. (18) in the population equation (8b), the number of households in the city is

$$\sum_{(x,y)\in\Theta}\left\{\left(\frac{\alpha+\beta}{\beta}\right)\frac{(1 - a(x,y) - z(x,y))l(x,y)}{y - f[d_{\mathrm{C}}(x,y)]}\, p^{\tau}(x,y)\right\} = \bar{N}. \tag{19}$$

The no arbitrage condition across neighborhoods means that utility is equal across all neighborhoods (5b), that is

$$\begin{aligned} &u(x,y) = u(s,r) \quad \text{for all}\,(x,y)\in\Theta \quad \text{and} \quad (s,r)\in\Theta,\\ &c_i(x,y)^{\alpha}h_i(x,y)^{\beta}A(x,y)^{\gamma} = c_i(s,r)^{\alpha}h_i(s,r)^{\beta}A(s,r)^{\gamma}. \end{aligned} \tag{20}$$

Plugging in for the composite good and housing and rearranging the terms Eq. (20) becomes

$$\frac{(y - f[d_{\mathrm{C}}(x,y)])^{\alpha+\beta}A(x,y)^{\gamma}}{p^{\tau}(x,y)^{\beta}} = \frac{(y - f[d_{\mathrm{C}}(s,r)])^{\alpha+\beta}A(s,r)^{\gamma}}{p^{\tau}(s,r)^{\beta}}. \tag{21}$$

Using Eq. (21) we solve for the price ratio:

$$\frac{p^{\tau}(s,r)}{p^{\tau}(x,y)} = \left(\frac{A(s,r)}{A(x,y)}\right)^{\gamma/\beta}\left(\frac{y - f[d_{\mathrm{C}}(s,r)]}{y - f[d_{\mathrm{C}}(x,y)]}\right)^{(\alpha+\beta)/\beta}. \tag{22}$$

Given the price ratio Eq. (22) for all $(x, y) \in \Theta$ and $(s, r) \in \Theta$ and Eq. (19) we solve for equilibrium prices:

$$p^{\tau}(x,y) = \max\left\{\frac{\beta N(v - f[d_{\mathrm{C}}(x,y)])^{(\alpha\pm\beta)/\beta}A(x,y)^{\gamma/\beta}}{(\alpha+\beta)\sum_{(s,r)\in\Theta}\left\{(1-a(x,y)-z(x,y))\bar{l}(x,y)(v - f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\right\}}, p_{\mathrm{g}}(x,y)\right\}.$$

Using the above equilibrium price equation we solve for housing consumption and the number of households in each neighborhood:

$$h_i(x,y) = \frac{\sum_{(s,r)\in\Theta}\left\{(1-a(x,y)-z(x,y))l(x,y)(v - f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\right\}}{N(v - f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}},$$

$$n(x,y) = \frac{N(1-a(x,y)-z(x,y))l(x,y)(v - f[d_{\mathrm{C}}(x,y)])^{\alpha/\beta}A(x,y)^{\gamma/\beta}}{\sum_{(s,r)\in\Theta}\left\{(1-a(x,y)-z(x,y))l(x,y)(v - f[d_{\mathrm{C}}(s,r)])^{\alpha/\beta}A(s,r)^{\gamma/\beta}\right\}}.$$

The allocation of the composite good is already given in the demand Eq. (10a) as

$$c_i(x,y) = \frac{\alpha}{\alpha+\beta}(v - f[d_{\mathrm{C}}(x,y)]).$$

Given the open space distribution $\{a(x,y)\}_{(x,y)\in\Theta}$ use the housing price $p^{\tau}(x,y)$ and demand $h_i(x,y)$ equations to solve for the fringe neighborhoods using the condition (6) from the definition of the equilibrium.

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